Z-TRANSFORM AND STIRLING NUMBERS

R. Sivaraman*, J. López-Bonilla & V. M. Salazar Del Moral****

* Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, Tamil Nadu, India

** ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

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Abstract:

We use the Z-transform to study the Stirling numbers of the second kind and their connection with identities showed by Spivey and He-Qi.

Key Words: Stirling Numbers of the Second Kind, Z-Transform.

1. Introduction:

It is immediate the Z-transform [1] of the sequence $\{\frac{a^r}{r!}\}$, r = 0, 1, 2,… :

$$Z\left\{\frac{a^r}{r!}\right\} = e^{a/z}\,, \qquad (1)$$

Then for $a = -1$:

$$Z\left\{\frac{(-1)^r}{r!}\right\} = e^{-1/z}\,. \qquad (2)$$

Now in (1) we apply the Euler-Grunert's operator $(a\frac{d}{da})^m$, $m \geq 0$[2-6] to obtain:

$$Z\left\{\frac{r^m a^r}{r!}\right\} = e^{a/z}\sum_{j=0}^{m} S_m^{[j]}\left(\frac{a}{z}\right)^j, \qquad (3)$$

Thus for $a = 1$:

$$Z\left\{\frac{r^m}{r!}\right\} = e^{1/z}\sum_{j=0}^{m} S_m^{[j]}\frac{1}{z^j}, \qquad (4)$$

Involving the Stirling numbers of the second kind [4, 7].

In Sec. [2] we use (2) and (4) to deduce the Spivey's identities [8] studied by He-Qi [9].

2. Spivey's Relations:

From (2) and (4):

$$Z\left\{\frac{A(n,m)}{n!}\right\} \coloneqq \sum_{j=0}^{\infty} S_m^{[j]}\frac{1}{z^j} = Z\left\{\frac{(-1)^q}{q!}\right\} Z\left\{\frac{r^m}{r!}\right\}, \qquad (5)$$

Then the sequence $\left\{\frac{A(n,m)}{n!}\right\} = \{S_m^{[n]}\}$ is the Cauchy convolution [10] of $\left\{\frac{(-1)^q}{q!}\right\}$ and $\{\frac{r^m}{r!}\}$, that is:

$$A(n,m) = n!\; S_m^{[n]} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m = \frac{(-1)^n}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^m, \qquad (6)$$

In agreement with the Spivey's identities [8, 9]:

$$A(n,m) = \begin{cases} 0, & m < n, \\ n!\,, & m = n, \\ (n+1)!\,\frac{n}{2}\,, & m = n+1. \end{cases} \qquad (7)$$

Remark:

In [6] is the following property:

$$Z\left\{\frac{1}{r!}\sum_{j=0}^{r}\frac{(-1)^j\, j!}{(1-\lambda)^{j+1}}S_r^{[j]}\right\} = (e^{1/z}-\lambda)^{-1}, \quad \lambda \neq 1, \qquad (8)$$

Which for $\lambda = 0$ implies (2). For the case $\lambda = 1$ we have the expression:

$$Z\left\{\frac{B_n}{n!}\right\} = \frac{1}{z}(e^{1/z}-1)^{-1}, \qquad (9)$$

Involving the Bernoulli numbers [3, 4], and if $\lambda = -1$:

$$Z\left\{(1-2^{n+1})\frac{B_{n+1}}{(n+1)!}\right\} = (e^{1/z}+1)^{-1}, \quad n = 0, 1, 2, \ldots \qquad (10)$$

References:

1. A. Grove, An introduction to the Laplace transform and the Z-transform, Prentice-Hall, London (1991).
2. J. Stopple, A primer of analytic number theory, Cambridge University Press (2003).
3. T. Arakawa, T. Ibukiyama, M. Kaneko, Bernoulli numbers and zeta functions, Springer, Japan (2014).
4. J. Quaintance, H. W. Gould, Combinatorial identities for Stirling numbers, World Scientific, Singapore (2016).
5. J. Yaljá Montiel-Pérez, J. López-Bonilla, S. Vidal-Beltrán, On the Melzak and Wilf identities, The J. of Middle East and North Africa Sci. 3, No. 11 (2017) 1-3.

6. R. Sivaraman, J. Bulnes, Laid Elkhiri, J. López-Bonilla, Sun's binomial inversion formulae, Euler's operator, and Z-transform, J. de Ciencia e Ingeniería 16, No. 2 (2024) 9-14.
7. V. Barrera-Figueroa, J. López-Bonilla, R. López-Vázquez, On Stirling numbers of the second kind, Prespacetime Journal 8, No. 5 (2017) 572-574.
8. M. Z. Spivey, The art of proving binomial identities, CRC Press, Boca Raton, Fl, USA (2019).
9. Chun-Ying He, Feng Qi, A unified proof of three combinatorial identities related to the Stirling numbers of the second kind, J. Inequal. Math. Anal. 2, No. 1 (2026) 1-6.
10. R. Sivaramakrishnan, Classical theory of arithmetic functions, Marcel Dekker, New York (1989).