



## Z-TRANSFORM AND STIRLING NUMBERS

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**Abstract:**

We use the Z-transform to study the Stirling numbers of the second kind and their connection with identities showed by Spivey and He-Qi.

**Key Words:** Stirling Numbers of the Second Kind, Z-Transform.

**1. Introduction:**

It is immediate the Z-transform [1] of the sequence  $\left\{ \frac{a^r}{r!} \right\}, r = 0, 1, 2, \dots :$

$$Z \left\{ \frac{a^r}{r!} \right\} = e^{a/z}, \tag{1}$$

Then for  $a = -1$ :

$$Z \left\{ \frac{(-1)^r}{r!} \right\} = e^{-1/z}. \tag{2}$$

Now in (1) we apply the Euler-Grunert's operator  $\left( a \frac{d}{da} \right)^m, m \geq 0$  [2-6] to obtain:

$$Z \left\{ \frac{r^m a^r}{r!} \right\} = e^{a/z} \sum_{j=0}^m S_m^{[j]} \left( \frac{a}{z} \right)^j, \tag{3}$$

Thus for  $a = 1$ :

$$Z \left\{ \frac{r^m}{r!} \right\} = e^{1/z} \sum_{j=0}^m S_m^{[j]} \frac{1}{z^j}, \tag{4}$$

Involving the Stirling numbers of the second kind [4, 7].

In Sec. [2] we use (2) and (4) to deduce the Spivey's identities [8] studied by He-Qi [9].

**2. Spivey's Relations:**

From (2) and (4):

$$Z \left\{ \frac{A(n,m)}{n!} \right\} := \sum_{j=0}^{\infty} S_m^{[j]} \frac{1}{z^j} = Z \left\{ \frac{(-1)^q}{q!} \right\} Z \left\{ \frac{r^m}{r!} \right\}, \tag{5}$$

Then the sequence  $\left\{ \frac{A(n,m)}{n!} \right\} = \left\{ S_m^{[n]} \right\}$  is the Cauchy convolution [10] of  $\left\{ \frac{(-1)^q}{q!} \right\}$  and  $\left\{ \frac{r^m}{r!} \right\}$ , that is:

$$A(n,m) = n! S_m^{[n]} = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m = \frac{(-1)^n}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} j^m, \tag{6}$$

In agreement with the Spivey's identities [8, 9]:

$$A(n,m) = \begin{cases} 0, & m < n, \\ n!, & m = n, \\ (n+1)! \frac{n}{2}, & m = n+1. \end{cases} \tag{7}$$

**Remark:**

In [6] is the following property:

$$Z \left\{ \frac{1}{r!} \sum_{j=0}^r \frac{(-1)^j j!}{(1-\lambda)^{j+1}} S_r^{[j]} \right\} = (e^{1/z} - \lambda)^{-1}, \quad \lambda \neq 1, \tag{8}$$

Which for  $\lambda = 0$  implies (2). For the case  $\lambda = 1$  we have the expression:

$$Z \left\{ \frac{B_n}{n!} \right\} = \frac{1}{z} (e^{1/z} - 1)^{-1}, \tag{9}$$

Involving the Bernoulli numbers [3, 4], and if  $\lambda = -1$ :

$$Z \left\{ (1 - 2^{n+1}) \frac{B_{n+1}}{(n+1)!} \right\} = (e^{1/z} + 1)^{-1}, \quad n = 0, 1, 2, \dots \tag{10}$$

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