International Journal of Current Research and Modern Education (IJCRME)
International Peer Reviewed - Refereed Research Journal, Website: www.crystalpen.in
Impact Factor: 6.925, ISSN (Online): 2455 - 5428, Volume 11, Issue 1, January - June, 2026

ON THE PIZIAK'S FORMULA FOR THE MOORE-PENROSE'S PSEUDOINVERSE

R. Sivaraman*, J. López-Bonilla & J. Muñoz-Bernabé****

* Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, Tamil Nadu, India

** ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5,1er. Piso, Col. Lindavista CP 07738, CDMX, México

Cite This Article: R. Sivaraman, J. López-Bonilla & J. Muñoz-Bernabé, "On the Piziak's Formula for the Moore-Penrose's Pseudoinverse", International Journal of Current Research and Modern Education, Volume 11, Issue 1, January - June, Page Number 107-108, 2026.

DOI: https://doi.org/10.5281/zenodo.20736401

Abstract:

We give an elementary proof of the Piziak's expression for the Moore-Penrose generalized inverse matrix.

Key Words: Characteristic Polynomial, Faddeev-Sominsky's Method, Moore-Penrose Pseudoinverse, Singular Value Decomposition.

1. Introduction:

The Faddeev-Sominsky's procedure [1-8] shows the importance of the matrices:

$$C_k = R^k + r_1 R^{k-1} + r_2 R^{k-2} + \ldots + r_{k-1} R + r_k I, \qquad k = 1, 2, \ldots, n, \tag{1}$$

Where the r_j are the coefficients of the characteristic polynomial of R[9-12]. Here we consider the Gram matrix:

$$R_{nxn} = A_{nxm} A^T_{mxn}, \tag{2}$$

Such that rank $A = p =$ rank R ;the matrix (2) is symmetric, non-defective and satisfies the relation:

$$R^{p+1} + r_1 R^p + \cdots + r_{p-1} R^2 + r_p R = 0\,. \tag{3}$$

On the other hand, the Singular Value Decomposition (SVD) of Agives the factorization [13-17]:

$$A = U_{nxp}\,\Lambda_{pxp}\,V^T_{mxp}, \qquad U^T U = V^T V = I_{pxp}, \qquad \Lambda = \mathrm{Diag}\left(\lambda_1, \ldots, \lambda_p\right), \tag{4}$$

Where $\lambda_j > 0,\ j = 1, 2, \ldots, p$ are the singular values of A, and the corresponding Moore-Penrose's pseudoinverse has the following expression [13, 18-24]:

$$A^+ = V\,\Lambda^{-1} U^T\,. \tag{5}$$

In Sec. 2 we employ (1) – (4) to obtain the Piziak's formula [25] for the Moore-Penrose's matrix.

2. Gram matrix and SVD:

From (2) and (4) is immediate the property:

$$R^j = U\,\Lambda^{2j} U^T, \qquad j \geq 1, \tag{6}$$

Then (3) implies the relation:

$$\Lambda^{2(p+1)} + r_1 \Lambda^{2p} + r_2 \Lambda^{2(p-1)} + \cdots + r_{p-1}\Lambda^4 + r_p \Lambda^2 = 0,$$

Whose multiplication by Λ^{-4} gives the condition:

$$\Lambda^{2(p-1)} + r_1 \Lambda^{2(p-2)} + r_2 \Lambda^{2(p-3)} + \cdots + r_{p-2}\Lambda^2 + r_{p-1} I + r_p \Lambda^{-2} = 0. \tag{7}$$

From (1), (6) and (7):

$$C_{p-1} = U\left(\Lambda^{2(p-1)} + r_1 \Lambda^{2(p-2)} + r_2 \Lambda^{2(p-3)} + \cdots + r_{p-2}\Lambda^2\right) U^T + r_{p-1} I,$$
$$= U\left(-r_{p-1} I - r_p \Lambda^{-2}\right) U^T + r_{p-1} I \qquad \therefore \qquad U^T C_{p-1} = -r_p \Lambda^{-2} U^T,$$

Hence $V\,\Lambda\,U^T C_{p-1} = -r_p V\,\Lambda^{-1} U^T$ where we can apply (4) and (5) to deduce the Piziak's formula [25] for the Moore-Penrose's pseudoinverse:

$$A^+ = -\frac{1}{r_p} A^T C_{p-1}\,. \tag{8}$$

References:

1. D. K. Faddeev, I. S. Sominsky, Collection of problems on higher algebra, Moscow (1949).
2. V. N. Faddeeva, Computational methods of linear algebra, Dover, New York (1959) Chap. 3.
3. D. K. Faddeev, Methods in linear algebra, W. H. Freeman, San Francisco, USA (1963).
4. B. Hanzon, R. Peeters, Computer algebra in systems theory, Dutch Institute of Systems and Control, Course Program 1999-2000.
5. J. Caltenco, J. López-Bonilla, R. Peña, Characteristic polynomial of A and Faddeev's method for A^{-1}, Educatia Matematica (Romania) 3, No. 1-2 (2007) 107-112.
6. J. Caltenco, J. López-Bonilla, R. Peña, On the Faddeev's technique to construct A^{-1}, Prespacetime J. 6, No. 11 (2015) 1230-1232.
7. J. López-Bonilla, H. Torres, S. Vidal-Beltrán, On the Faddeev-Sominsky's algorithm, World Scientific News 106 (2018) 238-244.
8. J. D. Bulnes, N. Islam, J. López-Bonilla, Leverrier-Takeno and Faddeev-Sominsky algorithms, Scientia Magna 17, No. 1 (2022) 77-84.

9. H. Takeno, A theorem concerning the characteristic equation of the matrix of a tensor of the second order, Tensor NS 3 (1954) 119-122.
10. J. H. Wilkinson, The algebraic eigenvalue problem, Clarendon Press, Oxford (1965).
11. G. Helmberg, P. Wagner, G. Veltkamp, On Faddeev-Leverrier's method for the computation of the characteristic polynomial of a matrix and of eigenvectors, Linear Algebra and its Applications 185 (1993) 219-233.
12. I. Guerrero-Moreno, J. López-Bonilla, J. Rivera-Rebolledo, Leverrier-Takeno coefficients for the characteristic polynomial of a matrix, J. Inst. Eng. (Nepal) 8, No. 1-2 (2011) 255-258.
13. C. Lanczos, Linear differential operators, Dover, New York (1997).
14. V. Gaftoi, J. López-Bonilla, G. Ovando, Singular value decomposition and Lanczos potential, in "Current Topics in Quantum Field Theory Research", Ed. O. Kovras, Nova Sci. Pub., New York (2007) 513-516.
15. G. Bahadur Thapa, P. Lam-Estrada, J. López-Bonilla, Singular factorization of an arbitrary matrix, J. of the Institute of Engineering (Nepal) 12, No. 1 (2016) 77-86.
16. J. López-Bonilla, R. López-Vázquez, S. Vidal-Beltrán, Some applications of the singular value decomposition, African J. Basic & Appl. Sci. 10, No. 4 (2018) 90-92.
17. J. Bisgard, Analysis and linear algebra: the singular value decomposition and applications", Student Math. Library 94, Am. Math. Soc., Providence, Rhode Island (2021).
18. E. H. Moore, On the reciprocal of the general algebraic matrix, Bull. Amer. Math. Soc. 26, No. 9 (1920) 394-395.
19. R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955) 406-413.
20. A. Albert, Regression and the Moore-Penrose pseudoinverse, Academic Press, New York (1972).

A. Ben-Israel, The Moore of the Moore-Penrose inverse, Electron. J. Linear Algebra 9 (2002) 150-157.

21. G. Bahadur-Thapa, P. Lam-Estrada, J. López-Bonilla, On the Moore-Penrose generalized inverse matrix, World Scientific News 95 (2018) 215-223.
22. O. Maria Baksalary, G. Trenkler, The Moore-Penrose inverse: a hundred years on a frontline of physics research, Eur. Phys. J. H 46, No. 9 (2021) 1-10.
23. S. Vidal-Beltrán, J. López-Bonilla, J. Yaljá Montiel-Pérez, G. Georgina Pérez-Sánchez, Moore-Penrose pseudo-inverse applied to the decoding of 5G and beyond (B5G) communication systems, Contemporary Maths, 6, No. 2 (2025) 1401-1413.
24. R. Piziak, The Frame algorithm and generalized inverses, Research Gate (2016), www.researchgate.net/publication/307559504